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LETTER TO THE EDITOR

Phase-space representation for Galilean quantum particles of arbitrary spin

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Abstract. The phase-space approach to quantisation is extended to incorporate spinning particles with Galilean symmetry. The appropriate phase space is the coadjoint orbit $\mathbb{R}^6 \times S^2$. From two basic principles, traciality and Galilean covariance, the Weyl symbol calculus is constructed. Then the Galilean-equivariant twisted products of functions on this phase space are identified.

In the conventional description, the states of a quantum-mechanical non-relativistic particle are identified with elements of the Hilbert space $\mathscr{H}^j = \mathbb{C}^{2j+1} \otimes L^2(\mathbb{R}^3, \mathrm{d}\xi)$. Since the pioneering work by Weyl [1], Wigner [2] and, foremost, Moyal [3] phase-space realisations of such a physical system, for $j = 0$, have attracted considerable attention. In this letter we extend the phase-space approach to cover spinning particles as well, within the framework of non-relativistic mechanics.

Let S^2 denote the manifold of states of a 'classical spin', i.e. the sphere. Let U^j be a physical (i.e. projective) unitary representation of the Galilei group G on $\mathscr{H}^j$, write $g \cdot A := U^j(g) A U^j(g^{-1})$ for any operator A on $\mathscr{H}^j$, and let $g \cdot u$ denote the action of G on the phase space $\mathbb{R}^6 \times S^2$, with coordinates $u := (\boldsymbol{q}, \boldsymbol{p}; \boldsymbol{n})$. By a 'Stratonovich–Weyl correspondence' we mean a rule assigning to every operator A a function W_A on the phase space $\mathbb{R}^6 \times S^2$, satisfying the following postulates.

(*a*) The correspondence is linear and one-to-one.

(*b*) Self-adjoint operators are mapped into real functions.

(*c*) The identity operator is mapped into the constant function 1.

(*d*) Traciality. For a suitable multiple $\mathrm{d}\mu^j$ of the ordinary measure on $\mathbb{R}^6 \times S^2$, the equation $\int W_A(u) W_B(u) \mathrm{d}\mu^j(u) = \operatorname{Tr} AB$ holds whenever both sides make sense.

(*e*) Covariance: $W_{g\cdot A}(u) = W_A(g^{-1} \cdot u)$ for $g \in \mathrm{G}$, $u \in \mathbb{R}^6 \times S^2$.

The problem of finding a Stratonovich–Weyl correspondence for a Galilean particle with arbitrary spin has an essentially unique solution. We collect first the necessary formulae for the Galilei group. Lévy-Leblond's notation [4] is employed throughout. A Galilean transformation, defined by $(b, \boldsymbol{a}, \boldsymbol{v}, R)(\boldsymbol{x}, t) := (R\boldsymbol{x} + \boldsymbol{v}t + \boldsymbol{a}, t + b)$, where $b \in \mathbb{R}$, $\boldsymbol{a}, \boldsymbol{v} \in \mathbb{R}^3$, $R \in \mathrm{SO}(3)$ and $(\boldsymbol{x}, t)$ are spacetime coordinates, has inverse $(b, \boldsymbol{a}, \boldsymbol{v}, R)^{-1} = (-b, R^{-1}(b\boldsymbol{v} - \boldsymbol{a}), -R^{-1}\boldsymbol{v}, R^{-1})$. It acts on phase space [5] by

$$(b, \boldsymbol{a}, \boldsymbol{v}, R) \cdot (\boldsymbol{q}, \boldsymbol{p}; n) := \left(R\left(\boldsymbol{q} - \frac{b}{m} p \right) + \boldsymbol{a} - b\boldsymbol{v}, R\boldsymbol{p} + m\boldsymbol{v}, R\boldsymbol{n} \right). \tag{1}$$

Here m is the mass of the particle.

It is well known that the projective representations of $\tilde{\mathrm{G}}$, the covering group of G, are obtained from linear representations of a 'splitting group' $\bar{\mathrm{G}}$ by passing to the quotient. The corresponding multiplier representations (which we will denote also by

U^j) act on $\mathscr{H}^j$ (which can be thought of as momentum space for the j-spin particle) by

$$[U^j(b, \boldsymbol{a}, \boldsymbol{v}, \tilde{R})\Phi]_s(\xi) := \exp\left[\frac{\mathrm{i}}{\hbar}\left(\frac{b|\xi|^2}{2m} - \boldsymbol{\xi}\cdot\boldsymbol{a} + \tfrac{1}{2}m\boldsymbol{a}\cdot\boldsymbol{v}\right)\right] \sum_{t=-j}^{j} \mathscr{D}^j_{st}(\tilde{R})\Phi_t(R^{-1}(\boldsymbol{\xi} - m\boldsymbol{v})) \tag{2}$$

where $\tilde{R} \in \mathrm{SU}(2)$, the covering group of SO(3), R is the rotation matrix corresponding to $\tilde{R}$ and the $\mathscr{D}^j_{st}(\tilde{R})$ are the usual matrix elements $\langle js|\pi_j(\tilde{R})|jt\rangle$ [6], where π_j denotes the irreducible representation of SU (2) on $\mathbb{C}^{2j+1}$, $s = -j, \ldots, j-1, j$.

The system of factors is

$$\omega_m(g, g') = \frac{1}{2\hbar}(-b'\boldsymbol{v}\cdot R\boldsymbol{v}' + \boldsymbol{v}\cdot R\boldsymbol{a}' - \boldsymbol{a}\cdot R\boldsymbol{v}')$$

if $g = (b, \boldsymbol{a}, \boldsymbol{v}, \tilde{R})$, $g' = (b', \boldsymbol{a}', \boldsymbol{v}', \tilde{R}')$ belong to $\tilde{\mathrm{G}}$. It restricts nicely to the exponent of the canonical commutation relations in Weyl form [1] on considering the subgroup of $\tilde{\mathrm{G}}$ of elements such that $b = 0$, $\tilde{R} = I$. We note that (1) comes naturally from Kirillov-Souriau theory [7, 8], as $\mathbb{R}^6 \times S^2$ is an orbit of the coadjoint action of $\tilde{\mathrm{G}}$, corresponding to a Casimir element $m > 0$.

By condition (a), we may write

$$W_A(u) = \mathrm{Tr}(A\Gamma^j(u))$$

for some operator-valued function Γ^j on $\mathbb{R}^6 \times S^2$. Now, by the tracial condition (d):

$$\begin{aligned}\mathrm{Tr}\, AB &= \int W_A(u)\, W_B(u)\mathrm{d}\mu^j(u) = \int \mathrm{Tr}(A\Gamma^j(u))\, W_B(u)\mathrm{d}\mu^j(u)\\ &= \mathrm{Tr}\left(A \int W_B(u)\Gamma^j(u)\mathrm{d}\mu^j(u)\right)\end{aligned}$$

which implies

$$B = \int W_B(u)\Gamma^j(u)\mathrm{d}\mu^j(u) \qquad \text{for any } B.$$

Thus the tracial condition, which is obviously imposed to assure the equality of standard quantum-mechanical and phase-space averages, has the important consequence that the correspondence $A \leftrightarrows W_A$ can be implemented with the *same* operator kernel Γ^j.

We now show that $\Gamma^j(u)$ is a tensor product of operators $\Delta^j(\boldsymbol{n})$ acting on $\mathbb{C}^{2j+1}$ and $\Pi(\boldsymbol{q}, \boldsymbol{p})$ acting on $L^2(\mathbb{R}^3, \mathrm{d}\boldsymbol{\xi})$: $\Gamma^j(\boldsymbol{q}, \boldsymbol{p}; \boldsymbol{n}) = \Delta^j(\boldsymbol{n}) \otimes \Pi(\boldsymbol{q}, \boldsymbol{p})$. Introduce the following functions over the sphere:

$$Z^j_{rs}(\boldsymbol{n}) := \frac{\sqrt{4\pi}}{2j+1} \sum_{l=0}^{2j} (2l+1)^{1/2} \langle jlr(s-r)|js\rangle\, Y_{l,s-r}(\boldsymbol{n}) \tag{3}$$

where Y_{lm} denotes the usual spherical harmonics and $\langle jlr(s-r)|js\rangle$ is a Clebsch-Gordan coefficient. Using the well known formula [6] for transforming spherical harmonics:

$$Y_{lm}(R\boldsymbol{n}) = \sum_{n=-l}^{l} \mathscr{D}^{l*}_{mn}(\tilde{R})\, Y_{ln}(\boldsymbol{n})$$

one derives [9], after some calculation,

$$Z^j_{rs}(R\boldsymbol{n}) = \sum_{p,q=u-j}^{j} \mathscr{D}^j_{rp}(\tilde{R})\mathscr{D}^{j*}_{sq}(\tilde{R})Z^j_{pq}(\boldsymbol{n}).$$

Define

$$\Delta^j(\boldsymbol{n}) := \sum_{r,s=-j}^{j} Z^j_{rs}(\boldsymbol{n})|jr\rangle\langle js|. \tag{4}$$

As $Z^j_{rs} = \bar{Z}^j_{sr}$, the Δ^j are self-adjoint. One computes easily that

$$\operatorname{Tr} \Delta^j(\boldsymbol{n}) = 1 \tag{5a}$$

$$\operatorname{Tr}(\Delta^j(\boldsymbol{m})\Delta^j(\boldsymbol{n})) = \frac{4\pi}{2j+1}\sum_{l=0}^{2j}\sum_{s=-l}^{l} Y_{ls}(\boldsymbol{m})Y^*_{ls}(\boldsymbol{n}) =: \frac{4\pi}{2j+1}K^j(\boldsymbol{m},\boldsymbol{n}). \tag{5b}$$

Here K^j is the reproducing kernel of the space of spherical harmonics of degree $\leqslant 2_j$. Now introduce

$$\Pi(\boldsymbol{q},\boldsymbol{p})\Phi(\boldsymbol{\xi}) := 2^3 \exp\left(\frac{2\mathrm{i}}{\hbar}\boldsymbol{q}\cdot(\boldsymbol{p}-\boldsymbol{\xi})\right)\Phi(2\boldsymbol{p}-\boldsymbol{\xi})$$

and compute [10, 11]

$$\operatorname{Tr}\Pi(\boldsymbol{q},\boldsymbol{p}) = 1 \tag{6a}$$

$$\operatorname{Tr}(\Pi(\boldsymbol{q},\boldsymbol{p})\Pi(\boldsymbol{q}',\boldsymbol{p}')) = (2\pi\hbar)^3\delta(\boldsymbol{q}-\boldsymbol{q}')\delta(\boldsymbol{p}-\boldsymbol{p}'). \tag{6b}$$

We note that the geometrical meaning of the $\Pi(\boldsymbol{q},\boldsymbol{p})$ as reflection operators was only uncovered some years ago by Grossmann [12] and Royer [13]. It is easily seen that the $\Pi(\boldsymbol{q},\boldsymbol{p})$ are self-adjoint.

From (5) and (6) it follows that the $\Gamma^j(u) = \Delta^j(\boldsymbol{n})\otimes\Pi(\boldsymbol{q},\boldsymbol{p})$ are self-adjoint and

$$\operatorname{Tr}\Gamma^j(u) = 1 \tag{7a}$$

$$\operatorname{Tr}(\Gamma^j(u)\Gamma^j(u')) = \frac{4\pi}{2j+1}(2\pi\hbar)^3\delta^j(u-u') \tag{7b}$$

with an obvious meaning for $\delta^j(u-u')$.

Now, our initial set of postulates is readily seen to translate into the following conditions for the family of operators $\Gamma^j(u)$:

(i) the $\Gamma^j(u)$ are self-adjoint;

(ii) $\int_{\mathbf{R}^6\times S^2}\Gamma^j(u)\mathrm{d}\mu^j(u) = I$;

(iii) $\int_{\mathbf{R}^6\times S^2}\operatorname{Tr}(\Gamma^j(u)\Gamma^j(u'))\Gamma^j(u')\mathrm{d}\mu^j(u') = \Gamma^j(u)$;

(iv) $\Gamma^j(g\cdot u) = U^j(g)\Gamma^j(u)U^j(g)^{-1}$, whenever $\tilde{g} = (b, \boldsymbol{a}, \boldsymbol{v}, \tilde{R}) \in \tilde{\mathrm{G}}$ and $g\cdot u$ is given by (1), with R being the rotation determined by $\tilde{R}\in \mathrm{SU}\ (2)$.

Taking

$$\mathrm{d}\mu^j(u) = (2\pi\hbar)^{-3}\frac{2j+1}{4\pi}\mathrm{d}\boldsymbol{q}\ \mathrm{d}\boldsymbol{p}\ \mathrm{d}\boldsymbol{n}$$

conditions (ii) and (iii) follow from (7). Now (iv) is verified by a direct calculation, using (2):

$$2^{-3}[U^j(b,\boldsymbol{a},\boldsymbol{v},\tilde{R})\Gamma^j(\boldsymbol{q},\boldsymbol{p};\boldsymbol{n})U^j((b,\boldsymbol{a},\boldsymbol{v},\tilde{R})^{-1})\Phi]_s(\boldsymbol{\xi})$$

$$=2^{-3}\exp\left[\frac{\mathrm{i}}{\hbar}\left(\frac{b|\boldsymbol{\xi}|^2}{2m}-\boldsymbol{\xi}\cdot\boldsymbol{a}+\tfrac{1}{2}m\boldsymbol{a}\cdot\boldsymbol{v}\right)\right]$$

$$\times\sum_{t=-j}^{j}\mathscr{D}^j_{st}(\tilde{R})(\Gamma^j(\boldsymbol{q},\boldsymbol{p};\boldsymbol{n})U^j((b,\boldsymbol{a},\boldsymbol{v},\tilde{R})^{-1}\Phi]_t(R^{-1}(\boldsymbol{\xi}-m\boldsymbol{v}))$$

$$=\exp\left[\frac{\mathrm{i}}{\hbar}\left(\frac{b|\boldsymbol{\xi}|^2}{2m}-\boldsymbol{\xi}\cdot\boldsymbol{a}+\tfrac{1}{2}m\boldsymbol{a}\cdot\boldsymbol{v}+2\boldsymbol{q}\cdot[\boldsymbol{p}-R^{-1}(\boldsymbol{\xi}-m\boldsymbol{v})]\right)\right]$$

$$\times\sum_{t,u=-j}^{j}\mathscr{D}^j_{st}(\tilde{R})Z^j_{tu}(\boldsymbol{n})[U^j((b,\boldsymbol{a},\boldsymbol{v},\tilde{R})^{-1})\Phi]_u(2\boldsymbol{p}-R^{-1}(\boldsymbol{\xi}-m\boldsymbol{v}))$$

$$=\exp\left[\frac{\mathrm{i}}{\hbar}\left(\frac{b|\boldsymbol{\xi}|^2}{2m}-\boldsymbol{\xi}\cdot\boldsymbol{a}+\tfrac{1}{2}m\boldsymbol{a}\cdot\boldsymbol{v}+2(R\boldsymbol{q})\cdot(R\boldsymbol{p}+m\boldsymbol{v}-\boldsymbol{\xi})\right.\right.$$

$$\left.\left.-\frac{b}{2m}|2R\boldsymbol{p}+m\boldsymbol{v}-\boldsymbol{\xi}|^2-(b\boldsymbol{v}-\boldsymbol{a})\cdot(2R\boldsymbol{p}+m\boldsymbol{v}-\boldsymbol{\xi})-\tfrac{1}{2}m(b\boldsymbol{v}-\boldsymbol{a})\cdot\boldsymbol{v}\right)\right]$$

$$\times\sum_{t,u,v=-j}^{j}\mathscr{D}^j_{st}(\tilde{R})Z^j_{tu}(\boldsymbol{n})\mathscr{D}^{j*}_{vu}(\tilde{R})\Phi_v(2R\boldsymbol{p}+2m\boldsymbol{v}-\boldsymbol{\xi})$$

$$=\exp\left\{\frac{\mathrm{i}}{\hbar}\left[2\left(R\boldsymbol{q}-\frac{b}{m}R\boldsymbol{p}+\boldsymbol{a}+b\boldsymbol{v}\right)\cdot(R\boldsymbol{p}+m\boldsymbol{v}-\boldsymbol{\xi}\right]\right\}$$

$$\times\sum_{v=-j}^{j}Z^j_{sv}(R\boldsymbol{n})\Phi_v(2R\boldsymbol{p}+2m\boldsymbol{v}-\boldsymbol{\xi})$$

$$=2^{-3}\left[\Gamma^j\left(R\left(\boldsymbol{q}-\frac{b}{m}\boldsymbol{p}\right)+\boldsymbol{a}-b\boldsymbol{v},R\boldsymbol{p}+m\boldsymbol{v},R\boldsymbol{n}\right)\Phi\right]_s(\boldsymbol{\xi}).$$

The conclusion is that there exists a phase-space representation for the description of a non-relativistic spinning particle, as a theory of 'Wigner functions' over $\mathbb{R}^6\times S^2$. Full details of such a theory for spin are given in [9]; there it is seen that the $Z^j_{ss}(\boldsymbol{n})$ are the Wigner functions corresponding to the states $|js\rangle$.

Remark 1. The family Γ^j is essentially unique: unicity of the Π comes from the Stone–von Neumann theorem; in the definition of the Δ^j a few sign changes could be made, but it can be shown that only the definition (3) and (4) make physical sense.

Remark 2. In the modern approach to phase-space quantum mechanics [14–18] the Stratonovich–Weyl correspondence is de-emphasised in favour of the *twisted product* of two functions on phase space, corresponding to the usual product of two operators. In that way the theory is formulated autonomously as a calculus of functions on phase space. The twisted product, denoted by $\times$, is determined by the condition that $W_A\times W_B=W_{AB}$ for all operators A, B. Using the Stratonovich–Weyl correspondence, we find

$$(f\times h)(u)=\int_{\mathbb{R}^6\times S^2}\int_{\mathbb{R}^6\times S^2}L^j(u,v,w)f(v)h(w)\,\mathrm{d}\mu^j(v)\,\mathrm{d}\mu^j(w)$$

where $L^j(u,v,w)=\mathrm{Tr}(\Gamma^j(u)\Gamma^j(v)\Gamma^j(w))$. For instance,

$$L^{1/2}(u, v, w) = 16(1+3\boldsymbol{n}\cdot\boldsymbol{n}'+3\boldsymbol{n}'\cdot\boldsymbol{n}''+3\boldsymbol{n}''\cdot\boldsymbol{n}+3\sqrt{3}\mathrm{i}[\boldsymbol{n}, \boldsymbol{n}', \boldsymbol{n}''])$$
$$\times\exp\left(\frac{2\mathrm{i}}{\hbar}(\boldsymbol{q}\cdot\boldsymbol{p}'-\boldsymbol{q}'\cdot\boldsymbol{p}+\boldsymbol{q}'\cdot\boldsymbol{p}''-\boldsymbol{q}''\cdot\boldsymbol{p}'+\boldsymbol{q}''\cdot\boldsymbol{p}-\boldsymbol{q}\cdot\boldsymbol{p}'')\right)$$

if $u=(\boldsymbol{q}, \boldsymbol{p}; \boldsymbol{n})$, $v=(\boldsymbol{q}', \boldsymbol{p}'; \boldsymbol{n}')$, $w=(\boldsymbol{q}'', \boldsymbol{p}''; \boldsymbol{n}'')$.

The tracial condition becomes

$$\int_{\mathbb{R}^6\times S^2}(f\times h)(u)\,\mathrm{d}\mu^j(u)=\int_{\mathbb{R}^6\times S^2}f(u)h(u)\,\mathrm{d}\mu^j(u)$$

and the covariance condition implies equivariance of the twisted product:

$$(f\times h)^g(u)=(f^g\times h^g)(u)\qquad\text{for all } g\in\tilde{\mathrm{G}} \tag{8}$$

where $f^g(u):=f(g^{-1}\cdot u)$. In fact, (8) is true for the larger group $\mathrm{Sp}(6;\mathbb{R})\rtimes\mathbb{R}^6$ of transformations of phase space (or its twofold covering $\mathrm{Mp}(6;\mathbb{R})\rtimes\mathbb{R}^6$, to be precise). The canonical generators of this group are 'distinguished' Hamiltonians, for which the quantum dynamics is rendered in the phase space essentially in classical terms.

Remark 3. Formulae (6) need some justification, as the operators Π are not of trace class. We intend to show elsewhere in the spirit of [19] that they hold in a distributional sense.

Remark 4. Generalisation of the formalism developed here to any finite number of particles is straightforward.

In summary, the Moyal phase-space formalism now provides a self-contained approach to non-relativistic quantum mechanics, including both spatial and spin variables, which is fully covariant under the Galilei group.

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